A simple approach to fuzzy critical path analysis in project networks

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Abstract— In this paper, we propose a new approach to critical path analysis in a project network whose activity times are uncertain. The uncertain parameters in the project network are represented by fuzzy numbers. We use fuzzy arithmetic and a fuzzy ranking method to determine the fuzzy critical path of the project network without converting the fuzzy activity times to classical numbers. The proposed method is compared with the existing method using examples.

Index Terms—Trapezoidal fuzzy numbers, Fuzzy arithmetic, Ranking, Project network, Critical path, Floats, Eearliest start, Eearliest finish, Latest start, Latest finish.

1 Introduction

ACTIVITY networks are highly useful for the performance evaluation of many types of projects. A constructed network is an important tool in the planning and control of actual project implementation. Project management is divided in to different subjects like scheduling, control, time management, resource management and cost management among which time management is more significant.

Critical path method is a network-based method designed for planning and managing of complicated projects in real world applications. The main purpose of critical path method is to evaluating project performance and to identifying the critical activities on the critical path so that the available resources could be utilized on these activities in the project network in order to reduce project completion time. With the help of the critical path, the decision maker can adopt a better strategy of optimizing the time and the available resources to ensure the earlier completion and the quality of the project.

The successful implementation of critical path method requires the availability of clear determined time duration for each activity. However in real life situations, project activities are subject to considerable uncertainty that may lead to numerous schedule disruptions. This uncertainity may arise from a number of possible sources like: activities may take more or less time than originally estimated, resources may become unavailable, material may arrive behind schedule, due dates may have to be changed, new activities may have to be incorporated or activities may have to be dropped due to changes in the project scope, weather conditions may cause severe delays, etc. A disrupted schedule incurs higher costs due to missed duedates and deadlines, resource idleness, higher work-in-process inventory and increased system nervousness due to frequent rescheduling. As a result, the conventional approaches, both deterministic and random process, tend to be less effective in conveying the imprecision or vagueness nature of the linguistic assessment. Consequently, the fuzzy set theory can play a significant role in this kind of decision making environment to tackle the unknown or the vagueness about the time duration of activities in a project network. To effectively deal with the ambiguities involved in the process of linguistic estimate times, the trapezoidal fuzzy numbers are used to characterize fuzzy measures of linguistic values.

Dubois et al [5] extended the fuzzy arithmetic operations to compute the latest starting time of each activity in a project network. Hapke et al. [7] used fuzzy arithmetic operations to compute the latest starting time of each activity in a project network. To find critical path in a fuzzy project network Yao et al. [11] used signed distance ranking of fuzzy numbers. Chen et al. [2] used defuzzification method to find possible critical paths in a fuzzy network.

Dubois et al [6] assigns a different level of importance to each activity on a critical path for a randomly chosen set of activities. To deal with completion time management and the critical degrees of all activities for a project network. C. T. Chen and S. F. Huang, applied fuzzy method for measuring criticality in project network. Ravi Shankar et al [10] proposed an analytical method for finding critical path in a fuzzy project network.

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In this paper, we propose a new method to find the critical path in a project network with out defuzzifying the fuzzy activity durations. The proposed method is based upon a new fuzzy arithmetic given in [3]. It makes project analysis in fuzzy environment more accurate. Finally illustrative numerical examples are given to demonstrate validity of the proposed method. The rest of this paper is organized as follows: In section 2, we recall the basic concept of fuzzy numbers, ranking and other related results. In section 3, we introduce fuzzy critical path analysis. In section 4 numerical examples are given to illustrate the theory.

2 PRELIMANARIES

The aim of this section is to present some notations, notions and results which are useful in our further consideration.

Definition 2.1

A fuzzy set $\tilde{a}$ defined on the set of real numbers R is said to be a fuzzy number if its membership function has the following characteristics:

(i). $\tilde{a}$ is convex, i.e.

$$\tilde{a}(\lambda x_1 + (1-\lambda)x_2) = \text{minimum}\{\tilde{a}(x_1), \tilde{a}(x_2)\},$$
$$\text{for all } x_1, x_2 \in R \text{ and } \lambda \in [0,1]$$

(*ii*). $\tilde{a}$ is normal i.e., there exists an $x_0 \in R$ such that $\tilde{a}(x_0) = 1$

(*iii*). $\tilde{a}$ is Piecewise continuous.

Definition 2.2

A fuzzy number $\tilde{a}$ in R is said to be a trapezoidal fuzzy number if its membership function $\tilde{a}: R \to [0,1]$ has the following characteristics:

$$\tilde{a} = \begin{cases} \dfrac{(x - a_1)}{(a_2 - a_1)}, & a_1 \le x \le a_2 \\ 1, & a_2 \le x \le a_3 \\ \dfrac{(x - a_4)}{(a_3 - a_4)}, & a_3 \le x \le a_4 \\ 0, & \text{otherwise} \end{cases}$$

We denote this trapezoidal fuzzy number by $\tilde{a} = (a_1, a_2, a_3, a_4)$
We use *F(R)* to denote the set of all trapezoidal fuzzy numbers.

For an interval $\tilde{a} = [a_1, a_2]$, define $\tilde{a}: R \to [0,1]$ as

$$\tilde{a}(x) = \begin{cases} 1, & \text{for all } x \in [a_1, a_2] \\ 0, & \text{otherwise} \end{cases}$$

Then $\tilde{a}$ is a fuzzy number. That is an interval can be viewed as a fuzzy number whose membership function takes value 1 over the interval and 0 anywhere else. Hence every closed and bounded interval is a special type of fuzzy number.

In a trapezoidal fuzzy number,

(i). if $a_1 = a_3$, then $\tilde{a}$ is known as a triangular fuzzy number.

(ii). if $a_1 = a_2$, $a_3 = a_4$, then $\tilde{a}$ is an interval fuzzy number or an interval number.

(iii). If $a_2 - a_1 = a_4 - a_3$, then $\tilde{a}$ is known as a symmetric trapezoidal fuzzy number.

2.2 [3] Arithmetic operations on trapezoidal fuzzy numbers

Let $\tilde{a} = (a_1, a_2, a_3, a_4)$ and $\tilde{b} = (b_1, b_2, b_3, b_4)$ be any two trapezoidal fuzzy numbers in *F(R)*. The arithmetic operations on $\tilde{a}$ *and* $\tilde{b}$ are defined as:

$$\tilde{a} * \tilde{b} = \{a_i * b_i / a_i \in \tilde{a}, b_i \in \tilde{b}\}, \text{ where } * \in \{+, -, \bullet, \div\}$$

In particular, for any two trapezoidal fuzzy numbers $\tilde{a} = (a_1, a_2, a_3, a_4)$ and $\tilde{b} = (b_1, b_2, b_3, b_4)$, we define

Addition:
$$\tilde{a} + \tilde{b} = (a_1, a_2, a_3, a_4) + (b_1, b_2, b_3, b_4) = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4)$$

Subtraction:
$$\tilde{a} - \tilde{b} = (a_1, a_2, a_3, a_4) - (b_1, b_2, b_3, b_4) = (a_1 - b_1, a_2 - b_2, a_3 - b_3, a_4 - b_4)$$

2.3 Ranking of trapezoidal fuzzy numbers

An efficient approach for comparing the fuzzy numbers is by the use of a ranking function $\Re : F(R) \to R$, where $F(R)$ is a set of fuzzy numbers defined on set of real numbers, which maps each fuzzy number into a real number, where a natural order exists i.e.,

For every $\tilde{a} = (a_1, a_2, a_3, a_4) \in F(R)$, the ranking function

$\Re : F(R) \to R$ defined as

$$\Re(\tilde{a}) = \left(\frac{a_1 + 2a_2 + 2a_3 + a_4}{6} \right)$$

For any two trapezoidal fuzzy numbers $\tilde{a} = (a_1, a_2, a_3, a_4)$ and $\tilde{b} = (b_1, b_2, b_3, b_4)$, we have the following comparison

$$\tilde{a} \succ \tilde{b} \text{ iff } \Re(\tilde{a}) > \Re(\tilde{b})$$
$$\tilde{a} \prec \tilde{b} \text{ iff } \Re(\tilde{a}) < \Re(\tilde{b})$$
$$\tilde{a} \approx \tilde{b} \text{ iff } \Re(\tilde{a}) = \Re(\tilde{b})$$

A trapezoidal fuzzy number $\tilde{a} = (a_1, a_2, a_3, a_4)$ is said to be positive if $\Re(\tilde{a}) > 0$

That is $\tilde{a} \succ \tilde{0}$ if $\Re(\tilde{a}) > 0$. Also $\tilde{a} \approx \tilde{0}$ *if* $\Re(\tilde{a}) = 0$ and $\tilde{a} \prec \tilde{0}$ if $\Re(\tilde{a}) < 0$. If $\tilde{a} \approx \tilde{b}$, then the trapezoidal fuzzy numbers $\tilde{a}$ and $\tilde{b}$ are said to be equivalent.

3 FUZZY CRITICAL PATH ANALYSIS

A fuzzy project network is an acyclic digraph, where the vertices represent events, and the directed edges represent the activities, to be performed in a project. We denote this fuzzy project network by $\tilde{N} = (\tilde{V}, \tilde{A}, \tilde{T})$.

Let $\tilde{V} = \{\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \ldots, \tilde{v}_n\}$ be the set of fuzzy vertices (events), where $\tilde{v}_1$ and $\tilde{v}_n$ are the tail and head events of the project, and each $\tilde{v}_i$ belongs to some path from $\tilde{v}_1$ to $\tilde{v}_n$. Let $\tilde{A} \in (\tilde{V} \times \tilde{V})$ be the set of directed edges $\tilde{A} = \{\tilde{a}_{ij} = (\tilde{v}_i, \tilde{v}_j) / \text{ for } \tilde{v}_i, \tilde{v}_j \in \tilde{V}\}$, that represents the activities to be performed in the project. Activity $\tilde{a}_{ij}$ is then represented by one, and only one, arrow with a tail event $\tilde{v}_i$ and a head event $\tilde{v}_j$. For each activity $\tilde{a}_{ij}$, a fuzzy number $\tilde{t}_{ij} \in \tilde{T}$ is defined as the fuzzy time required for the completion of $\tilde{a}_{ij}$. A critical path is a longest path from the initial event $\tilde{v}_1$ to the terminal event to $\tilde{v}_n$ of the project network, and an activity $\tilde{a}_{ij}$ on a critical path is called a critical activity.

3.1 Notations

$\tilde{t}_{ij}$: The fuzzy activity time of activity $\tilde{a}_{ij}$

$\tilde{ES}_j$: The earliest fuzzy time of event $\tilde{v}_j$

$\tilde{LS}_i$: The latest fuzzy time of event $\tilde{v}_i$

$\tilde{T}_{ij}$: The total float of fuzzy activity $\tilde{a}_{ij}$

P_i: The i-th path of the fuzzy project network.

P: The set of all paths in a fuzzy project network.

CPM (P_k): The fuzzy completion time of path P_k in a fuzzy project network.

Property 1 If $\tilde{a}_{ij} = (\tilde{v}_i, \tilde{v}_j)$, $\tilde{a}_{mn} = (\tilde{v}_m, \tilde{v}_n)$ are two fuzzy activities, activity $\tilde{a}_{ij}$ is a predecessor of activity $\tilde{a}_{mn}$ iff there is a chain from event j to event m in project network.

Property 2 If $\tilde{a}_{ij} = (\tilde{v}_i, \tilde{v}_j)$, $\tilde{a}_{mn} = (\tilde{v}_m, \tilde{v}_n)$ are two fuzzy activities, activity $\tilde{a}_{ij}$ is an immediate predecessor of activity $\tilde{a}_{mn}$ iff either j = m, or there exists a chain from event *j* to event *m* in the project network consisting of dummy activities only.

Property 3 $$CPM(P_k) = \sum_{\substack{1 \le i < j \le n \\ i, j \in P_k}} \tilde{T}_{ij}, \quad P_K \in P$$

Definition 3.1 Assume that there exists a path PC in a fuzzy project network such that CPM (PC) = min {CPM (P_i) | $P_i \in P$}, then the path PC is a fuzzy critical path.

Theorem 1 Assume that the fuzzy activity times of all activities in a project network are trapezoidal fuzzy numbers, then there exists fuzzy critical path in the project network

3.1 Algorithm for Fuzzy critical activity

Let $\tilde{ES}_i$ and $\tilde{LS}_i$ be the earliest fuzzy event time, and the latest fuzzy event time for event *i*, respectively. Functions that define the earliest starting times, latest starting times and floats in terms of fuzzy activity durations are convex, normal whose membership functions are piecewise continuous, hence the quantities such as earliest fuzzy event time $\tilde{ES}_i$, the latest fuzzy event time $\tilde{LS}_i$ and the floats $\tilde{T}_i$ are also trapezoidal fuzzy numbers for an event *i* respectively.

Step 1: Identify Fuzzy activities in a fuzzy project

Step 2: Establish precedence relationships of all fuzzy activities by applying a fuzzy ranking function.

Step 3: Construct the fuzzy project network with trapezoidal fuzzy numbers as fuzzy activity times

Step 4: Let $\tilde{ES}_1$ be the earliest fuzzy event time and $\tilde{LS}_1$ be the latest fuzzy event time for the initial event $\tilde{v}_1$ of the project network and assume that $\tilde{ES}_1 = \tilde{LS}_1 = \tilde{0}$. Compute the earliest fuzzy event time $\tilde{ES}_j$ of the event $\tilde{v}_j$ by using the formula

$$\tilde{ES}_j = \max_{i \in N: i \to j} \{\tilde{ES}_i + \tilde{t}_{ij}\} \quad (1)$$

Step 5: Let $\tilde{ES}_n$ be the earliest fuzzy event time and $\tilde{LS}_n$ be the latest fuzzy event time for the terminal event $\tilde{v}_n$ of the fuzzy project network and assume that $\tilde{ES}_n = \tilde{LS}_n$. Compute the latest fuzzy event time $\tilde{LS}_i$ by using the following equation

$$\tilde{LS}_i = \min_{i \in N} \{\tilde{LS}_j - \tilde{t}_{ij}\} \quad (2)$$

Step 6: Compute the total float $\tilde{T}_{ij}$ of each fuzzy activity $\tilde{a}_{ij}$ by using the following equation

$$\tilde{T}_{ij} = \{\tilde{LS}_j - \tilde{ES}_i - \tilde{t}_{ij}\} \quad (3)$$

Hence we can obtain the earliest fuzzy event time, latest fuzzy event time, and the total float of every fuzzy activity by using equations (1), (2) and (3).

Step 7: If $\tilde{T}_{ij} = \tilde{0}$, then the activity $\tilde{a}_{ij}$ is said to be a *Fuzzy critical activity.* That is activities with zero total float are called *Fuzzy critical activities*, and are always found on one or more *Fuzzy critical paths*.

Step 8: The length of the longest *Fuzzy critical path* from the start of the fuzzy project to its finish is the minimum time required to complete the Fuzzy Project. This (or these) Fuzzy critical path(s) determine the minimum fuzzy project duration.

4 NUMERICAL EXAMPLES

In this section fuzzy project network problems are presented to demonstrate the computational process of fuzzy critical path analysis proposed above.

Example-1:
Suppose that there is a project network with the set of fuzzy events $\tilde{V} = \{1,2,3,4,5\}$, the fuzzy activity time for each activity is shown in Table 1. All the durations are in hours.

Table 1: Activity duration of each activity in a fuzzy project network

Activity	Activity duration
1-2	(10,15,15,20)
1-3	(30,40,40,50)
2-3	(30,40,40,50)
1-4	(15,20,25,30)
2-5	(60,100,150,180)
3-5	(60,100,150,180)
4-5	(60,100,150,180)

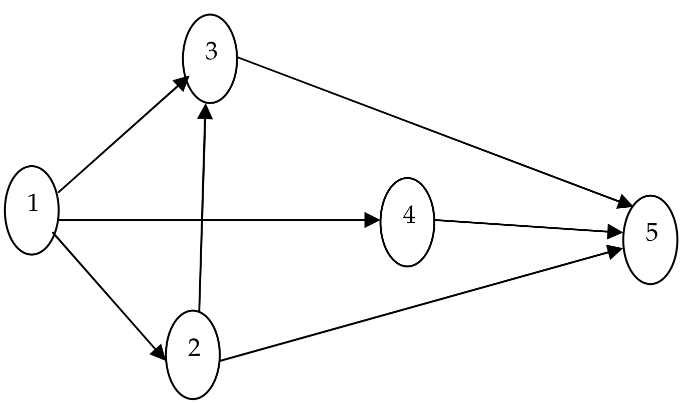

Figure 1: Fuzzy project network-I

Fuzzy critical path is $1 \rightarrow 2 \rightarrow 3 \rightarrow 5$
The minimum fuzzy project duration is the length of the fuzzy critical path.

The fuzzy project duration is (100, 155, 205, 250) fuzzy hours.

Table 2: Calculation of total float for each activity in a fuzzy project network and critical path

	Duration	Earliest Start	Earliest Finish	Latest Start	Latest finish	Total Float
(1-2)	(10,15,15,20)	(0,0,0,0)	(10,15,15,20)	(0,0,0,0)	(10,15,15,20)	**(0,0,0,0)**
(1-3)	(30,40,40,50)	(0,0,0,0)	(30,40,40,50)	(10,15,15,20)	(40,55,55,70)	(10,15,15,20)
(1-4)	(15,20,20,30)	(0,0,0,0)	(15,20,25,30)	(25,35,30,40)	(40,55,55,70)	(25,35,30,40)
(2-3)	(30,40,40,50)	(10,15,15,20)	(40,55,55,70)	(10,15,15,70)	(40,55,55,70)	**(0,0,0,0)**
(2-5)	(60,100,150,180)	(10,15,15,20)	(70,115,165,200)	(30,35,40,50)	(100,115,205,250)	(30,40,40,50)
(3-5)	(60,100,150,180)	(40,55,55,70)	(100,155,205,250)	(40,55,55,70)	(100,155,205,250)	**(0,0,0,0)**
(4-5)	(60,110,150,180)	(15,20,25,30)	(75,130,175,210)	(25,25,30,40)	(100,155,205,250)	(25,25,30,40)

Example-2:

Suppose that there is a project network, as Figure 2, with the set of events $\tilde{V} = \{1,2,3,4,5,6\}$, the fuzzy activity time for each activity as shown in Table 3. All the durations are in hours.

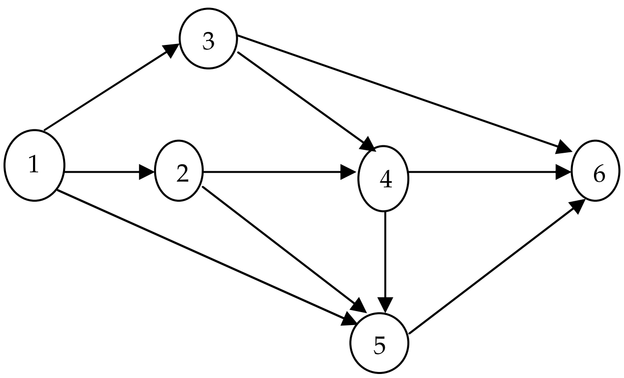

Figure 2: Fuzzy project network-II

Table 3: Fuzzy activity duration of each activity in Example 2.

Fuzzy activity	Fuzzy activity duration
(1-2)	(2,2,3,4)
(1-3)	(2,3,3,6)
(1-5)	(2,3,4,5)
(2-4)	(2,2,4,5)
(2-5)	(2,4,5,8)
(3-4)	(1,1,2,2)
(3-6)	(7,8,11,15)
(4-5)	(2,3,3,5)
(4-6)	(3,3,4,6)
(5-6)	(1,1,1,2)

Fuzzy critical path is $1 \rightarrow 3 \rightarrow 6$

The minimum fuzzy project duration is the length of the fuzzy critical path.

The fuzzy project duration is (9, 11, 14, 21) fuzzy hours.

Table 4: Calculation of total float for each activity in a fuzzy project network and critical path

Activity	Duration	Earliest Start	Earliest Finish	Latest Start	Latest finish	Total Float
(1-2)	(2,2,3,4)	(0,0,0,0)	(2,2,3,4)	(2,3,3,5)	(4,5,6,9)	(4,5,6,9)
(1-3)	(2,3,3,6)	(0,0,0,0)	(2,3,3,6)	(0,0,0,0)	(2,3,3,6)	**(0,0,0,0)**
(1-5)	(2,3,4,5)	(0,0,0,0)	(2,3,4,5)	(6,7,9,14)	(8,10,13,19)	(6,7,9,14)
(2-4)	(2,2,4,5)	(2,2,3,4)	(4,4,7,9)	(4,5,6,9)	(6,7,10,14)	(2,3,3,5)
(2-5)	(2,4,5,8)	(2,2,3,4)	(4,6,8,12)	(6,6,8,11)	(8,10,13,19)	(4,4,5,7)
(3-4)	(1,1,2,2)	(2,3,3,6)	(3,4,5,8)	(5,6,8,12)	(6,7,10,14)	(3,3,5,6)
(3-6)	(7,8,11,15)	(2,3,3,6)	(9,11,14,21)	(2,3,3,6)	(9,11,14,21)	**(0,0,0,0)**
(4-5)	(2,3,3,5)	(4,4,7,9)	(6,7,10,14)	(6,7,10,14)	(8,10,13,19)	(2,3,3,5)
(4-6)	(3,3,4,6)	(4,4,7,9)	(7,7,11,15)	(6,8,10,15)	(9,11,14,21)	(2,4,3,6)
(5-6)	(1,1,1,2)	(6,7,10,14)	(7,8,11,16)	(8,10,13,19)	(9,11,14,21)	(2,3,3,3)

4 Conclusion

This paper proposes an algorithm to tackle the problem in fuzzy project analysis. The validity of the proposed method is examined with numerical example.

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